INSTRUCTIONS: Give a complete solution to each problem and show all work in the space below the problem. Partial credit is possible. You may use your scientific calculator. Graphing and programmable calculators are not allowed. Books and notes are not allowed.

1. (20 points) Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = < y, z, x > \) and the space curve \( C \) is described by \( \mathbf{r} = < \cos t, \sin t, 1 - \cos t > \) with \( 0 \leq t \leq 2\pi \).

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C < \sin t, 1 - \cos t, \cos t > \cdot < -\sin t, \cos t, \sin t > \, dt
\]

\[
= \int_0^{2\pi} ( -\sin^2 t + \cos t - \cos^2 t + \sin t \cos t ) \, dt
\]

\[
= \left. -t + \sin t + \frac{1}{2} \sin^2 t \right|_0^{2\pi}
\]

\[= -2\pi\]
2. (20 points) Use the Fundamental Theorem of Line Integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$\mathbf{F} = \langle yz, xz, xy + 2z \rangle$ and $\mathbf{r} = \langle 1 + 2t, 4t, -2 + 7t \rangle$ with $0 \leq t \leq 1$.

Let $f = f(x, y, z) = xyz + z^2$

Notice that $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle yz, xz, xy + 2z \rangle = \mathbf{F}$

When $t = 0$, $\mathbf{r} = \langle 1, 0, -2 \rangle$

and when $t = 1$, $\mathbf{r} = \langle 3, 4, 5 \rangle$

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(3, 4, 5) - f(1, 0, -2) \]

\[ = [(3 \times 4 \times 5) + 5^2] - [(1 \times 0 \times -2) + (-2)^2] \]

\[ = 60 + 25 - 0 - 4 \]

\[ = 81 \]
3. (20 points) Use Green’s Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = \langle \sqrt{x+y^2}, x + \sqrt{y} \rangle \) and \( C \) is the line segment given by \( y = 0 \) from \((0,0)\) to \((\pi,0)\) followed by the arc of the curve \( y = \sin x \) from \((\pi,0)\) to \((0,0)\).

Let \( P = \sqrt{x} + y^2 \) and \( Q = x + \sqrt{y} \)

Clearly, \( \mathbf{F} = \langle P, Q \rangle \) and \( d\mathbf{r} = \langle dx, dy \rangle \)

**Therefore,**

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_P dP + Qdy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

\[
= \int_0^\pi \int_0^{\sin x} \left( 1 - 2y \right) \, dy \, dx
\]

\[
= \int_0^\pi \left[ y - y^2 \right]_{y=0}^{y=\sin x} \, dx
\]

\[
= \int_0^\pi \left( \sin x - \sin^2 x \right) \, dx
\]

\[
= \int_0^\pi \left( \sin x - \frac{1}{2} + \frac{1}{2} \cos 2x \right) \, dx
\]

\[
= -\cos x - \frac{1}{2} x + \frac{1}{4} \sin 2x \bigg|_0^\pi
\]

\[
= -(-1) - \frac{1}{2} \pi + (1)
\]

\[
= 2 - \frac{\pi}{2}
\]
4. (20 points) Find the surface area of the part of the plane given by \( 4x + 8y + z = 16 \) that lies in the first octant.

\[
\begin{align*}
A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \\
&= \iint_D \sqrt{1 + (-4)^2 + (-8)^2} \, dA \\
&= \iint_D \sqrt{81} \, dA \\
&= 9 \iint_D \, dA \\
&= 9A = 9(4) = 36
\end{align*}
\]

OR

\[
= 9 \int_0^4 \int_0^{2-\frac{1}{2}x} \, dy \, dx = \ldots = 36
\]
5. (20 points) Evaluate \( \iint_S \mathbf{F} \cdot d\mathbf{S} \) where \( \mathbf{F} = \langle y, x, 2xy \rangle \) and \( S \) is the part of the paraboloid \( z = 4 - (x^2 + y^2) \) that lies above the rectangle \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 2 \).

\[
\mathbf{F} = \langle y, x, 2xy \rangle = \langle P, Q, R \rangle \quad \text{implies that} \quad P = y, \quad Q = x, \quad \text{and} \quad R = 2xy
\]

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( -P \frac{\partial P}{\partial x} - Q \frac{\partial Q}{\partial y} + R \right) \, dA
\]

\[
= \iint_D \left[ -y(-2x) - x(-2y) + 2xy \right] \, dA
\]

\[
= \int_0^1 \int_0^2 4xy \, dy \, dx
\]

\[
= 4 \left( \int_0^1 x \, dx \right) \left( \int_0^2 y \, dy \right)
\]

\[
= 4 \left( \frac{1}{2} x^2 \bigg|_0^1 \right) \left( \frac{1}{2} y^2 \bigg|_0^2 \right)
\]

\[
= 4 \left( \frac{1}{2} \right) (2)
\]

\[
= \boxed{4}
\]